# HEAT CONDUCTION IN A SERIES OF INHOMOGENEOUS SLABS

O. V. Golubeva and Sh. V. Verdikhanov

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A general theory is worked out for boundary-value problems of steady-state, two-dimensional heat conduction in a series of inhomogeneous media. Specific examples are discussed.

The theory of steady-state heat conduction is of considerable interest because of its practical applications in several situations in modern technology. A characteristic feature of heat-conducting media is their inhomogeneity; because of this complication, comparatively few papers have appeared on this subject (noteworthy among these papers are [1-4]). In the present paper we continue this work.

1. Steady-state heat conduction in an isotropic medium is described by the Fourier law and the continuity equation:

$$\mathbf{j} = -k\nabla T, \qquad \operatorname{div} \mathbf{j} = 0,$$

where k is the (generally variable) heat-conduction coefficient, a measure of the inhomogeneity of the medium.

For the case of two-dimensional heat conduction in slabs with curvilinear surfaces, these equations lead to [5]

$$j_x = -\frac{k}{c} \cdot \frac{\partial T}{\partial x} = -\frac{k}{cp} \cdot \frac{\partial \psi}{\partial y}; \qquad j_y = -\frac{k}{c} \cdot \frac{\partial T}{\partial y} = \frac{k}{cp} \cdot \frac{\partial \psi}{\partial x}, \tag{1.1}$$

where x, y is the isothermal coordinate system for the surface $[ds^2 = c^2(dx^2 + dy^2)]$, h is the slab thickness, $p = kh$ is the thermal conductivity of the slab, T is the temperature, and $\psi$ is the heat-flux function.

From Eqs. (1.1) we find self-adjoint elliptic equations for T and $\psi$:

$$\frac{\partial}{\partial x}\left(p\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial T}{\partial y}\right) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{p} \cdot \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{p} \cdot \frac{\partial \psi}{\partial y}\right) = 0. \tag{1.3}$$

We assume that the heat-flux region $\sigma$ contains no singularities of the slab, i.e., that $p(x, y) \neq 0, \infty$, for $(x, y) \in \sigma$; and we assume that boundary conditions of the first kind are specified at the boundary of the region. Then for Eq. (1.2) the condition

$$T(x, y)|_L = T_L \tag{1.4}$$

corresponds to the specification of the temperature profile at the boundary of the heat-flux region, and for Eq. (1.3) the condition

$$\psi(x, y)|_L = \psi_L \tag{1.5}$$

corresponds to the specification of the heat flux across this region [5].

2. These boundary-value problems are solved by means of a Green's function. We consider a circle $L_1$ of radius $\varepsilon > 0$ which lies entirely within region $\sigma$ and is centered at point $x_0$, $y_0$. We use the first fundamental solution $T_{1f}$ of Eq. (1.2) [6]:

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$$T_{1f} = C\left[f_1(x,\ y,\ x_0,\ y_0)\ln\frac{1}{r} + f_2(x,\ y,\ x_0,\ y_0)\right], \tag{2.1}$$

where C is a constant and where $f_1$, $f_2$ are functions which, along with their partial derivatives, are continuous in the doubly connected region D, bounded by contours L and $L_1$,

$$f_1(x,\ y,\ x_0, y_0) = 1, \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

We introduce the function $T'_{1f}$, which satisfies Eq. (1.2) and which takes on the following value at boundary L:

$$T'_{1f}|_L = T_{1f}.$$

Then the Green's function of the boundary-value problem of the first kind for Eq. (1.2) is

$$G = \frac{1}{C}(T'_{1f} - T_{1f}), \quad (G|_L = 0). \tag{2.2}$$

Using the operator $\Delta(p,\ T)$ [see (1.2)], we write the generalized Green's function [5]

$$\iint_D [T\Delta(p,\ T_1) - T_1\Delta(p,\ T)]\,dxdy = \oint_{L_1} p\left(T\frac{\partial T_1}{\partial n_1} - T_1\frac{\partial T}{\partial n_1}\right)ds_1 + \oint_L p\left(T\frac{\partial T_1}{\partial n} - T_1\frac{\partial T}{\partial n}\right)ds. \tag{2.3}$$

Assuming that $T_1 = G$, T satisfies Eq. (1.2) and the boundary condition in (1.4), and also using the relations

$$\frac{\partial}{\partial n_1} = -\frac{\partial}{\partial r}, \quad r = \varepsilon, \quad ds_1 = \varepsilon d\varphi,$$

on circle $L_1$, we find the following from the generalized Green's function:

$$\varepsilon\int_0^{2\pi} p\left(T\frac{\partial G}{\partial r} - G\frac{\partial T}{\partial r}\right)_{r=\varepsilon} d\varphi = \oint_L p_L T_L\left(\frac{\partial G}{\partial n}\right)_L ds.$$

Then taking the limit $\varepsilon \to 0$, we find

$$T(x_0,\ y_0) = \frac{1}{2\pi\, p(x_0,\ y_0)}\oint_L p_L T_L\left(\frac{\partial G}{\partial n}\right)_L ds. \tag{2.4}$$

Equation (2.4) thus can be used to determine the temperature distribution within the region of variable thermal conductivity on the basis of the temperature profile at the boundary, if we know Green's function (2.2).

Analogously, we can construct the heat flux function $\psi$ within the region on the basis of its boundary values, once we have the Green's function corresponding to Eq. (1.3).

The Green's function in (2.2), or the first fundamental solution in (2.1), which is related to this Green's function, is known for only a limited number of functions $p(x, y)$, so that we should like to analyze the possibility of constructing solutions for a set of similar boundary-value problems for various functions p on the basis of the solution of one such problem. Methods for constructing such solutions are called "transfer methods" [5].

3. We assume that we have a solution for some boundary-value problem for T under boundary condition (1.4) in a slab with thermal conductivity p; then the function $\psi = T$ is the solution of the boundary-value problem for $\psi$ in a slab with thermal conductivity $1/p$ under boundary condition (1.5). In the region $\sigma$, the streamlines in slab p are rotated through an angle of $\pi/2$ in slab $1/p$ [5]. Slabs p and $1/p$ are "adjoint slabs," and their relationship can be written symbolically as

$$T[p] = \psi\left[\frac{1}{p}\right]. \tag{3.1}$$

4. Equations (1.1) are covariant with respect to conformal mappings; in particular, the isothermal surface coordinates x, y can be thought of as the coordinates of a plane onto which the base of the slab is mapped [5].

It follows that the solution of this boundary-value problem for T (or $\psi$) in slab $p(x,y)$ corresponds to an infinite set of solutions of the boundary-value problems in the slabs $p^*(\xi,\ \eta)$, which are the conformal mappings of the original slab. The variables $\xi$, $\eta$ define the plane or surface onto which plane z is conformally mapped.

By using a conformal-mapping procedure we can thus work from a single two-dimensional boundary-value problem to find a series of analogous problems in heat-conducting slabs which form a conformal family. This family can be written symbolically as

$$T\left[p(x,\ y)\right]=T\left[p(x(\xi,\ \eta),\ y(\xi,\ \eta))\right],\quad \frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial y},\quad \frac{\partial\xi}{\partial y}=-\frac{\partial\eta}{\partial x}. \tag{4.1}$$

5. Using a transformation of the function T of the type [6]

$$T=\frac{\Phi}{\sqrt{p}}, \tag{5.1}$$

we can write Eq. (1.2) in the canonical form

$$\Delta\Phi+\frac{\Delta\sqrt{p}}{\sqrt{p}}\,\Phi=0. \tag{5.2}$$

With given $p=p_1$ and, correspondingly, known $\Delta\sqrt{p_1}/\sqrt{p_1}=K_1$, we can evidently examine the family of different slabs satisfying the equation [7]

$$\Delta\sqrt{p}+K_1\sqrt{p}=0; \tag{5.3}$$

in this entire family of slabs the temperature distribution is governed by Eq. (5.3).

If $T_1$ and T are concrete and arbitrary solutions of Eq. (1.2) with $p=p_1$, then

$$T_1=\frac{\Phi_1}{\sqrt{p_1}},\quad T\left[p_1\right]=\frac{\Phi}{\sqrt{p_1}}. \tag{5.4}$$

Since $\Phi_1^2$ can be thought of as the law governing the behavior of the thermal conductivity of the slab, we can write

$$T\left[\Phi_1^2\right]=\frac{\Phi}{\Phi_1}. \tag{5.5}$$

From (5.1) and (5.4) we find

$$T\left[T_1^2p_1\right]=\frac{T\left[p_1\right]}{T_1}, \tag{5.6}$$

where $T_1$ is some particular solution of heat-conduction equation (1.2).

Slabs whose thermal conductivities satisfy Eq. (5.3) are "slabs of the $K_1$ series."

To transform from the temperature distribution in one slab of the series to another, we use Eq. (5.6). If the boundary-value problem of the first kind has been solved in slab $p_1$, then the analogous problems in the slabs of the $K_1$ series are governed by (5.6), with the following condition satisfied at boundary L:

$$T\left[T_1^2p_1\right]\big|_L=\frac{T\left[p_1\right]_L}{T_{1L}}. \tag{5.7}$$

The simplest series of slabs is $K_1=0$, for which the laws governing the behavior of the thermal conductivity are the squares of the harmonic functions u. We call such a series a "harmonic series." Since it includes $u_1=1$, we find, using (5.6),

$$T\left[u^2\right]=\frac{T\left[1\right]}{u}. \tag{5.8}$$

The problem of heat propagation in a harmonic series of slabs thus reduces to that in a homogeneous slab (p=1).

A particular property of the series $u^2$ is that it corresponds to a conformal family. Different boundary-value problems can be solved, however, by using (4.1) and (5.6).

6. We consider the temperature distribution in a slab of a harmonic series whose region $\sigma$, which contains no singularities of the slab, is bounded by circle L in the case of a conformal mapping onto the x, y plane.

We introduce the auxiliary polar coordinates $\rho$, $\varphi$:

$$x = x_1 + \rho\cos\varphi, \quad y = y_1 + \rho\sin\varphi, \tag{6.1}$$

where $x_1$, $y_1$ is the center of circle L. If R is the radius of this circle, then the equation of circle L is

$$\rho = R. \tag{6.2}$$

The first fundamental solution (2.1) for a homogeneous heat-conducting slab, $T_{1f}$, is

$$T_{1f}[1] = C\ln\frac{1}{r}, \quad \text{where } r = \sqrt{\rho^2 + \rho_0^2 - 2\rho_0\rho\cos(\varphi - \varphi_0)}. \tag{6.3}$$

Using the method of [8] for mapping singularities, we can write $T_{1f}$ in this case as

$$T'_{1f}[1] = C\ln\frac{1}{r_1}, \tag{6.4}$$

where

$$r_1 = \sqrt{R^2 + \frac{\rho_0^2\rho^2}{R^2} - 2\rho_0\rho\cos(\varphi - \varphi_0)}.$$

Obviously, $T_{1f}[1]|_L = T'_{1f}[1]|_L$. Hence the known Green's function G[1] for a homogeneous slab bounded by a circle can be written

$$G[1] = \frac{1}{C}\{T'_{1f}[1] - T_{1f}[1]\} = \ln\frac{r}{r_1}. \tag{6.5}$$

Returning to (5.8), we write the Green's function for the slabs of a harmonic series, with condition (6.2) at the boundary of the region, as

$$G[u^2] = \frac{u(x_0, y_0)}{u(x, y)}\ln\frac{r}{r_1}. \tag{6.6}$$

Since

$$T_{1f}[u^2] = C\frac{u(x_0, y_0)}{u(x, y)}\ln\frac{1}{r}, \quad T'_{1f}[u^2] = C\frac{u(x_0, y_0)}{u(x, y)}\ln\frac{1}{r_1},$$

then the Green's function in (6.6) describes the temperature distribution in a circular region of a harmonic family resulting from a heat source of strength $2\pi$ lying at the point $x_0 = x_1 + \rho_0\cos\varphi_0$, $y_0 = y_1 + \rho_0\sin\varphi_0$, with a vanishing temperature at the boundary of the region (6.2).

Conformal mappings of the x, y plane onto planes or curvilinear surfaces with the coordinates $\xi$, $\eta$ have the result that (6.6) describes, in terms of the variables $\xi$, $\eta$, the temperature distribution resulting from a heat source in region $\sigma_1$ (into which $\sigma$ is mapped) when the temperature vanishes at the boundary.

The Green's function in (6.6) can be used to solve the boundary-value problem of two-dimensional heat conduction with a circular boundary if the thermal conductivity of slab $u^2$ is the square of a harmonic function. Going back to Eq. (2.4), we write it for this case as

$$T(x_0, y_0) = \frac{R}{2\pi(x_0, y_0)}\int_0^{2\pi}\left(uT\frac{\partial}{\partial\rho}\cdot\frac{\ln r - \ln r_1}{u}\right)_{\rho=R} d\varphi. \tag{6.7}$$

With $p = u^2 = y^2$ and the boundary condition $T_{\rho=R} = 1$ for $0 < \varphi < \pi$ and $T_{\rho=R} = 0$ for $\pi < \varphi < 2\pi$ at the circle, we find, working from (6.7),

$$T[y^2] = \frac{1}{2\pi(y_1 + \rho_0\sin\varphi_0)}\left\{\frac{R^2 - \rho_0}{2\rho_0}\left(\cos\varphi\ln\frac{R^2 + \rho_0^2 + 2R\rho_0\cos\varphi_0}{R^2 + \rho_0^2 - 2R\rho_0\cos\varphi_0}\right.\right.$$
$$\left.\left. - \pi\sin\varphi_0\right) + \left(2\pi y_1 + \frac{\pi(R^2 + \rho_0^2)}{\rho_0}\sin\varphi_0\right)\left(1 - \frac{1}{\pi}\operatorname{arctg}\frac{R^2 - \rho_0^2}{2R\rho_0\sin\varphi_0}\right)\right\}, \tag{6.8}$$

which holds for $0 < \varphi < \pi$, and

$$T[y^2] = \frac{1}{2\pi(y_1 + \rho_0\sin\varphi_0)}\left\{\frac{R^2 - \rho_0^2}{2\rho_0}\left(\cos\varphi_0\ln\frac{R^2 + \rho_0 + 2R\rho_0\cos\varphi_0}{R^2 + \rho_0^2 - 2R\rho_0\cos\varphi_0}\right.\right.$$
$$\left. - \pi\sin\varphi_0\right) - \operatorname{arctg}\frac{R^2 - \rho_0^2}{2R\rho_0\sin\varphi_0}\left(2y_1 + \frac{R^2 + \rho_0^2}{\rho_0}\sin\varphi_0\right), \tag{6.8}$$

for $\pi < \varphi < 2\pi$. Using (5.6) and (5.7) and the solution (6.8) for this problem, we can find the solutions of analogous problems. For example, setting $T_1 = x$, we find

$$T[x^2y^2] = \frac{T[y^2]}{x}. \tag{6.9}$$

Substituting $T[y^2]$ from (6.8) into (6.9), we find equations for the temperature distribution on a circular plate whose thermal conductivity varies according to $x^2y^2$, with the boundary conditions

$$T|_{\rho=R} = \frac{1}{x_1 + R\cos\varphi} \quad \text{for} \quad 0 < \varphi < \pi,$$

$$T|_{\rho=R} = 0 \quad \text{for} \quad \pi < \varphi < 2\pi.$$

We note in conclusion that quite a large number of problems of two-dimensional heat conduction can now be solved, and work is being carried out on them.

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